PERGAMON

International Journal of Solids and Structures 37 (2000) 1561–1576

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Direct calculation of buckling strength of imperfect structures

Baisheng Wu*

Mechanics Section, Department of Mathematics, Jilin University, Changchun 130023, P.R. China

Received 26 February 1998; in revised form 20 October 1998

Abstract

This paper is concerned with the direct calculation of buckling strength of an imperfect structure. The main assumptions are that, for the perfect structure, there is a symmetric or a linear fundamental path and the corresponding first critical point is a simple symmetry-breaking or a simple bifurcation point. An extended system of limit points is proposed for which the newly introduced scaling parameters are regular solution and thus standard methods can be used to compute them. Using the extended system, one can directly obtain the exact buckling loads without tracing the postbuckling paths. An efficient implementation of Newton's method solving the extended system is presented and numerical examples are given. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

The load-bearing capacity of certain structures is highly sensitive to imperfections to which the geometries and materials are all subjected. Thin shells are the most prominent members of this class of imperfection-sensitive structures, which also includes certain types of columns, trusses, frames, arches, and thin-walled structures (Koiter, 1945; Thompson and Hunt, 1973). Small imperfections in these structures are inevitable and may result in a very significant deterioration of their buckling strength.

Up to now, three approaches have been proposed for the calculation of buckling strength of imperfect structures. Approach (i) is based on numerically tracking the equilibrium path of an imperfect structure beyond the point of maximum load (limit point). This can be achieved by a change in the 'loading' parameter (see e.g., Riks, 1979). This method is very useful and also available in most general purpose nonlinear codes, but it requires a separate analysis for each amplitude and shape of imperfection to be considered. Also equilibrium paths may have strong curvatures in the vicinity of bifurcation points, rendering them difficult to track numerically. Approach (ii) uses the Liapunov–Schmidt decomposition

* Fax: 0086 431 8923 907.
E-mail address: bs_wu@hotmail.com (B. Wu)

PII: S0020-7683(98)00311-4

together with Koiter's asymptotic expansion about the bifurcation point (see e.g., Casciaro et al., 1992; Peek and Kheyrkhahan, 1993). The implementations of the approach in finite element analysis, however, involves high order derivatives of the energy functional; also the range of validity of this method is restricted since such results are often based on the lower-order asymptotic analyses. Approach (iii) is to perturb the equilibrium and critical state equations simultaneously (see e.g., Thompson and Hunt, 1973; Godoy et al., 1995; Wu and Wang, 1997). This approach however, has the drawbacks similar to that of approach (ii).

The direct method has been developed in recent years and is a main tool numerically computing various kinds of nonlinear singular points (see e.g., Seydel, 1979; Werner and Spence, 1984; Wriggers and Simo, 1990; Eriksson, 1994; Wu, 1995). The method is based on an extended system which eliminates the singularity of original problem by introducing properly new equations. The goal of this paper is to introduce a direct method for calculating the buckling strength of imperfect structures. This method can allow simultaneous treatment of geometric and material imperfections, determine the limit point loads and the corresponding displacements exactly without tracking the equilibrium paths of imperfect structures, permit results to be implemented in a FEM program. The main assumptions are that, for the perfect structure, there is symmetric or a linear fundamental path and the corresponding first critical point is a simple symmetry-breaking or a simple bifurcation point.

Using the information on the bifurcation point and introducing some new scaling parameters, we construct an extended system to build a bridge between the bifurcation point of the perfect structure and the limit point of postbuckling path of the imperfect structure. Here, these scaling parameters have their origin in the Lyapunov–Schmidt–Koiter approach (see, e.g., Koiter, 1945; Potier-Ferry, 1987; Triantafyllidis and Peek, 1992) and the perturbation expansion technique (Wu and Wang, 1997). The solution to the extended system can be regularized by ξ, the amplitude of projection of incremental displacement u measured from the fundamental path on the normalized buckling mode u_{1c} of the perfect structure. By analyzing the stability of the obtained solutions, we can choose the direction of the continuation parameter ξ so that the limit point is the maximum value of the load on the corresponding post-buckling path. Finally, without tracking the equilibrium paths of the imperfect structures, one can directly find the exact limit loads and the corresponding displacements by continuing ξ with the extended system. The implementation of Newton's method solving the extended system is discussed. Three examples are used to illustrate the validity and applicability of the proposed method.

2. Basic formulae

In this paper, the term 'imperfect structure' will be used repeatedly and will denote a more detailed structural model, which simulates some of all of the unintended deviations of the real structure from the perfect model. These unintended deviations will be collectively denoted as imperfections. It is understood that the imperfections have been normalized so that, for zero amplitude of the imperfection modes, the imperfect structure is reduced to the perfect one.

Consider a discrete imperfect structure. Let the potential energy of the imperfect structure be given by $V(u, \lambda, w)$ where u denotes the additional displacement of the imperfect structure from its initial configuration, λ the loading parameter, and w the imperfection of the structure. Let R^n and W denote, respectively, spaces of displacement and imperfection of the structure. We introduce two inner products in the two spaces denoted by (u, v) for $u, v \in R^n$ and $[w_1, w_2]$ for $w_i \in W$ $(i = 1, 2)$ and the corresponding norms are $\|u\| = (u, u)^{1/2}$ for $u \in R^n$ and $|w| = [w, w]^{1/2}$ for $w \in W$, respectively. It is convenient to distinguish between the imperfection amplitude $\varepsilon = |w|$ and the normalized imperfection mode $\bar{u} = w/|w|$. The equilibrium equation of the imperfect structure is derived by using the stationary principle:

$$V_u(u, \lambda, \varepsilon\bar{u}) = 0. \tag{1}$$

Throughout the paper, we will use the following notations: various functional (Fréchet) derivatives of the potential energy with respect to u and w are denoted by subscript $(\ )_u$ and $(\ )_w$, respectively, etc.

We consider a critical point of the equilibrium path determined by eqn (1) that governs load-bearing capacity of the imperfect structure. The critical point is determined by (Thompson and Hunt, 1973; Budiansky, 1974)

$$V_{uu}(u, \lambda, \varepsilon\bar{u})\phi = 0 \tag{2}$$

for some eigenmode $\phi \in R^n, \phi \neq 0$. Consequently, for a given structural imperfection mode $\bar{u}$, the corresponding buckling load for the imperfection amplitude ε can be found by solving the system of nonlinear eqns (1) and (2) (in unknowns u, ϕ and λ) subject to the condition that the equilibrium position determined by the solution of eqns (1) and (2) is associated with the maximum value of load on the corresponding postbuckling path.

We assume that, for the perfect structure, there exists a fundamental equilibrium path $u = u_0(\lambda)$ as the load increases from zero, i.e.

$$V_u\big(u_0(\lambda), \lambda, 0\big) = 0. \tag{3}$$

Let $\lambda = \lambda_c$ be the buckling load for the perfect structure and it is assumed to be simple with corresponding buckling mode u_{1c} normalized by $\|u_{1c}\| = 1$. In mathematical terms:

$$V^c_{uu}u_{1c} = 0 \tag{4}$$

where superscript c denotes the corresponding derivatives of potential energy function V calculated at $(u, \lambda, w) = (u_0(\lambda_c), \lambda_c, 0)$. We further assume that, when load parameter λ passes increasingly through its critical value λ_c, the fundamental equilibrium path becomes unstable from stable (Thompson and Hunt, 1973; Budiansky, 1974; E1 Naschie, 1990):

$$V^c_{uu\lambda}u^2_{1c} =: \frac{d}{d\lambda}V_{uu}\big(u_0(\lambda), \lambda, 0\big)u^2_{1c}|_{\lambda=\lambda c} < 0. \tag{5}$$

We also assume that the effect of imperfections is of the first order (Thompson and Hunt, 1973; Budiansky, 1974; E1 Naschie, 1990; Ikeda and Murota, 1990):

$$V^c_{uw}u_{1c}\bar{u} \neq 0. \tag{6}$$

3. Calculation of the buckling strength via extended system

A method for calculating the buckling strength of imperfect structures is introduced in this section. The method is based on an extended system for which Newton-type algorithms have the quadratic rate of asymptotic convergence.

3.1. Formulation of extended system

Based on the Lyapunov–Schmidt–Koiter approach (Koiter, 1945; Potier-Ferry, 1987; Triantafyllidis and Peek, 1992) and the perturbation expansion technique (Wu and Wang, 1997), the displacement field u, eigenmode ϕ, load parameter λ and imperfection amplitude ε associated with the limit point of the imperfect structure can be written as $u = u_0(\lambda_c + \xi\Lambda) + \xi u_{1c} + \xi^2 v$, $\phi = u_{1c} + \xi\psi$, $\lambda = \lambda_c + \xi\Lambda$, $\varepsilon = \xi^2\tau$ where v and ψ satisfy $(u_{1c},v) = 0$ and $(u_{1c}, \psi) = 0$, respectively. We can therefore propose the following extended system for determining limit point of the imperfect structure:

$$E(v, \psi, \Lambda, \tau, \xi) := \begin{Bmatrix} f_1(v, \psi, \Lambda, \tau, \xi) \\ f_2(v, \psi, \Lambda, \tau, \xi) \\ (u_{1c}, v) \\ (u_{1c}, \psi) \end{Bmatrix} = 0 \tag{7a}$$

where

$$f_1 := \begin{cases} \dfrac{V_u\big(u_0(\lambda_c + \xi\Lambda) + \xi u_{1c} + \xi^2 v, \lambda_c + \xi\Lambda, \xi^2\tau\bar{u}\big)}{\xi^2}, & \text{if } \xi \neq 0, \\ V^c_{uu} v + \Lambda V^c_{uu\lambda} u_{1c} + \tau V^c_{uw}\bar{u} + \dfrac{1}{2} V^c_{uuu} u^2_{1c}, & \text{if } \xi = 0, \end{cases} \tag{7b}$$

$$f_2 := \begin{cases} \dfrac{V_{uu}\big(u_0(\lambda_c + \xi\Lambda) + \xi u_{1c} + \xi^2 v, \lambda_c + \xi\Lambda, \xi^2\tau\bar{u}\big)(u_{1c} + \xi\psi)}{\xi}, & \text{if } \xi \neq 0, \\ V^c_{uu}\psi + \Lambda V^c_{uu\lambda} u_{1c} + V^c_{uuu} u^2_{1c}, & \text{if } \xi = 0 \end{cases} \tag{7c}$$

and

$$E:\ R^n \times R^n \times R \times R \times R \rightarrow R^n \times R^n \times R \times R. \tag{7d}$$

In eqn (7a), the first formula describes the equilibrium condition, the second formula expresses the critical condition (limit point) of the equilibrium, while the third and fourth formulas indicate, respectively, orthogonal conditions resulting from the Liapunov–Schmidt decomposition. By using the Taylor expansion about $\xi = 0$ in the respective numerator in the first and second formulas in eqn (7a), the equations of the extended system in the case $\xi = 0$ can be obtained. For the stable solution of a nonlinear equation, using the Newton's method, say, it is important that the solution is nonsingular (Werner and Spence, 1984). The appearance of denominators ξ^2 and ξ, respectively, in the first and second formulas in eqn (7a) guarantees that the solutions to system (7) is nonsingular. This will be proven in the theorem below.

Consider the following system of equations:

$$E(v, \psi, \Lambda, \tau, 0) = 0. \tag{8}$$

It has the solution

$$(v, \psi, \Lambda, \tau) = \big(v_0, \psi_0, \Lambda_0, \tau_0\big) \tag{9a}$$

where

$$v_0 := v_{00} - (u_{1c}, v_{00})u_{1c},$$

$$\psi_0 := \psi_{00} - \big(u_{1c}, \psi_{00}\big)u_{1c},$$

$$\Lambda_0 := -\frac{V^c_{uuu}u^3_{1c}}{V^c_{uu\lambda}u^2_{1c}},$$

$$\tau_0 := \frac{V^c_{uuu}u^3_{1c}}{2V^c_{uw}u_{1c}\bar{u}} \tag{9b}$$

and v_{00} and ψ_{00} are, respectively, particular solutions to the following equations

$$V^c_{uu}v + \Lambda_0 V^c_{uu\lambda}u_{1c} + \tau_0 V^c_{uw}\bar{u} + \frac{1}{2}V^c_{uuu}u^2_{1c} = 0 \tag{10a}$$

and

$$V^c_{uu}\psi + \Lambda_0 V^c_{uu\lambda}u_{1c} + V^c_{uuu}u^2_{1c} = 0. \tag{10b}$$

Theorem. Let potential energy function $V(u, \lambda, \varepsilon\bar{u})$ satisfy conditions (3)–(6). Then, for $\xi = 0$, extended system (7) has a solution $(v_0, \psi_0, \Lambda_0, \tau_0)$ and its linearization with respect to (v, ψ, Λ, τ), at this solution, is non-singular.

Proof. Based on conditions (3) and (4), the homogeneous system associated with the linearization of extended system (7) with respect to (v, ψ, Λ, τ) at $\xi = 0$ has the form

$$V^c_{uu}\delta v + \delta\Lambda V^c_{uu\lambda}u_{1c} + \delta\tau V^c_{uw}\bar{u} = 0, \tag{11a}$$

$$V^c_{uu}\delta\psi + \delta\Lambda V^c_{uu\lambda}u_{1c} = 0, \tag{11b}$$

$$(u_{1c}, \delta v) = 0, \tag{11c}$$

$$(u_{1c}, \delta\psi) = 0. \tag{11d}$$

Taking inner product with u_{1c} in the two sides of eqns (11a) and (11b), respectively, and using eqn (4), one achieves

$$\delta\Lambda V^c_{uu\lambda}u^2_{1c} + \delta\tau V^c_{uw}u_{1c}\bar{u} = 0, \quad \delta\Lambda V^c_{uu\lambda}u^2_{1c} = 0. \tag{12}$$

Under conditions (5) and (6), one derives $\delta\Lambda = \delta\tau = 0$.

Substituting $\delta\Lambda = \delta\tau = 0$ into eqns (11a) and (11b), respectively, one has

$$V^c_{uu}\delta v = 0, \quad V^c_{uu}\delta\psi = 0 \tag{13}$$

and their solutions are given by

$$\delta v = c^1 u_{1c}, \quad \delta\psi = c^2 u_{1c} \tag{14}$$

where c^1 and c^2 are some real constants. Substitution of (14) into eqns (11c, d) leads to $c^1 = c^2 = 0$, hence

$\delta v = 0, \quad \delta\psi = 0.$

This completes the proof.

We can draw important conclusions from the Theorem.

Corollary. Let the conditions of the Theorem be satisfied. Then there exists a locally smooth solution branch $(v(\xi), \psi(\xi), \Lambda(\xi), \tau(\xi)) \in R^n \times R^n \times R \times R$ of $E(v, \psi, \Lambda, \tau, \xi) = 0$ such that $v(0) = v_0, \psi(0) = \psi_0, \Lambda(0) = \Lambda_0$ and $\tau(0) = \tau_0$, and for $\xi \neq 0, \lambda_s := \lambda_c + \xi\Lambda(\xi)$ is the buckling load for the imperfection amplitude $\xi^2\tau(\xi)$ when $\xi\Lambda(\xi) < 0$.

Proof. The existence of local smooth solutions $(v(\xi), \psi(\xi), \Lambda(\xi), \tau(\xi))$ satisfying the initial conditions follows from the regularity of the Jacobian of system $E = 0$ [eqn (7)] with respect to (v, ψ, Λ, τ) at $\xi = 0$.

On the other hand, for $\xi \neq 0$, from eqns (7a–c) one derives

$$V_u(u_0(\lambda_c + \xi\Lambda(\xi)) + \xi u_{1c} + \xi^2 v(\xi), \lambda_c + \xi\Lambda(\xi), \xi^2\tau(\xi)\bar{u}) = 0,$$

$$V_{uu}(u_0(\lambda_c + \xi\Lambda(\xi)) + \xi u_{1c} + \xi^2 v(\xi), \lambda_c + \xi\Lambda(\xi), \xi^2\tau(\xi)\bar{u})(u_{1c} + \xi\psi(\xi)) = 0. \quad (15)$$

Hence, for a given imperfection mode $\bar{u}$, $(u, \phi, \lambda) = (u_0(\lambda_c + \xi\Lambda(\xi)) + \xi u_{1c} + \xi^2 v(\xi), u_{1c} + \xi\psi(\xi), \lambda_c + \xi\Lambda(\xi))$ corresponds to a solution to the system of eqns (1) and (2) for the imperfection amplitude $\varepsilon = \xi^2\tau(\xi)$. By using the conclusion of stability analysis in Wu and Wang (1997), we know that, only if $\xi\Lambda(\xi) < 0, \lambda_s = \lambda_c + \xi\Lambda(\xi)$ is the buckling load (the maximum value of load) and the corresponding displacement $u_s := u_0(\lambda_c + \xi\Lambda(\xi)) + \xi u_{1c} + \xi^2 v(\xi)$. The proof is completed.

Note that our extended system (7) applies to both of asymmetric and symmetric bifurcation points. This is due to the nonsingularity of the solution to $E(v, \psi, \Lambda, \tau, 0) = 0$ has no relation with the classification of simple bifurcation points. Concerning the items to be implemented in the classification, they depend on problems considered, we refer readers to Casciaro et al. (1992). These implementations need not to be completed in our numerical method. This point will be explained in Section 3.3.

3.2. Procedure in calculations

Now we can propose a procedure for the calculation of the buckling strength of imperfect structures:

1. Determine the buckling load $\lambda = \lambda_c$ and buckling mode u_{1c} (normalized by $\|u_{1c}\| = 1$) of the corresponding perfect structure.
2. Extended system (7) (i.e. $E = 0$) can then be used to obtain the buckling load $\lambda_c + \xi\Lambda(\xi)$ and corresponding displacement $u_0(\lambda_c + \xi\Lambda(\xi)) + \xi u_{1c} + \xi^2 v(\xi)$ for the imperfection amplitude $\xi^2\tau(\xi)$ by continuing ξ from $\xi = 0$ along the direction of ξ satisfying $\xi\Lambda(\xi) < 0$.

Remark. The regularity of $(v, \psi, \Lambda, \tau) = (v_0, \psi_0, \Lambda_0, \tau_0)$ as the solution to extended system (7) for $\xi = 0$ guarantees our approach in step 2 can succeed.

If the perfect structure has a linear fundamental path, $u_0(\lambda) := \lambda u^f$ is easily obtained by solving a linear system with unknown u^f. Furthermore λ_c and u_{1c} can be achieved by solving an eigenvalue problem (Casciaro et al., 1992; Godoy et al., 1995)

$$K(\lambda)u = 0, \quad (u, u) = 1 \tag{16}$$

where $K(\lambda)$ is the tangent stiffness matrix along the linear fundamental path and $(u, u) = 1$ describes the normalized conditions for the buckling mode. An iteration algorithm for the solution to system (16) has been discussed (Casciaro et al., 1992). An alternative algorithm to obtain λ_c and u_{1c} is to apply the Newton's method to system (16). The implementation of solving the system is similar to that discussed in Section 3.3. After the buckling load and corresponding buckling mode are achieved, extended system (7) can be directly applied to get the buckling strength of imperfect structures. The implementation of solving the system in similar to that of a symmetric fundamental path which will later be described in detail in Section 3.3.

We next discuss the case that the perfect structure has a symmetric fundamental path. Let $g \rightarrow T_g$ is a unitary representation of a group Γ on the space R^n, such that $V(u, \lambda, 0)$ is invariant under Γ in the sense that

$$V(T_g u, \lambda, 0) = V(u, \lambda, 0), \quad \forall g \in \Gamma, \quad \forall u \in R^n. \tag{17}$$

Then it can be shown that $V_u(u, \lambda, 0)$ is covariant under Γ in the sense that (Troger and Steindl, 1991)

$$T_g V_u(u, \lambda, 0) = Vu(T_g u, \lambda, 0), \quad \forall g \in \Gamma, \quad \forall u \in R^n. \tag{18}$$

Let U_0 be an invariant subspace of R^n in the sense

$$U_0 = \{u \in R^n | T_g u = u, \quad \forall g \in \Gamma\}. \tag{19}$$

A symmetric fundamental path $u_0(\lambda)$ means

$$u_0(\lambda) \in U_0, \quad \forall \lambda \geqslant 0. \tag{20}$$

Suppose that the first critical point on the symmetric fundamental path is a simple symmetry-breaking bifurcation point ($u_{1c} \notin U_0$). Then λ_c, $u^c := u_0(\lambda_c)$ and u_{1c} can be obtained by solving a corresponding extended system given by Werner and Spence (1984) or Werner (1984).

3.3. Implementation of solving the extended system

In the case that the perfect structure has a symmetric fundamental path, we discuss the implementation of solving extended system (7). For this case, extended system (7) can be transformed to a more available form:

$$G(u, v, \psi, \Lambda, \tau, \xi) := \left\{ \begin{array}{c} V_u(u, \lambda_c + \xi\Lambda, 0) \\ g_1(u, v, \psi, \Lambda, \tau, \xi) \\ g_2(u, v, \psi, \Lambda, \tau, \xi) \\ (u_{1c}, v) \\ (u_{1c}, \psi) \end{array} \right\} = 0 \tag{21a}$$

where

$$g_1 := \frac{V_u(u + \xi u_{1c} + \xi^2 v, \lambda_c + \xi\Lambda, \xi^2 \tau \bar{u})}{\xi^2}, \tag{21b}$$

$$g_2 := \frac{V_{uu}\big(u + \xi u_{1c} + \xi^2 v,\, \lambda_c + \xi\Lambda,\, \xi^2\tau\bar{u}\big)(u_{1c} + \xi\psi)}{\xi} \tag{21c}$$

and

$$G: \quad U_0 \times R^n \times R^n \times R \times R \times R \to U_0 \times R^n \times R^n \times R \times R. \tag{21d}$$

Set $n_0 = \dim U_0$. In general, it will be easily to identify U_0 with R^{n_0} using the isomorphism $I_0 : U_0 \to R^{n_0}$, $I_0(u) = u^s$. Then extended system (21) becomes

$$H\big(u^s, v, \psi, \Lambda, \tau, \xi\big) := \begin{Bmatrix} I_0 V_u\big(I_0^{-1}u^s, \lambda_c + \xi\Lambda, 0\big) \\ g_1\big(I_0^{-1}u^s, v, \psi, \Lambda, \tau, \xi\big) \\ g_2\big(I_0^{-1}u^s, v, \psi, \Lambda, \tau, \xi\big) \\ (u_{1c}, v) \\ (u_{1c}, \psi) \end{Bmatrix} = 0 \tag{22a}$$

where

$$H: \quad R^{n0} \times R^n \times R^n \times R \times R \times R \to R^{n0} \times R^n \times R^n \times R \times R. \tag{22b}$$

System (22) can be solved by Newton's method. In the continuation with respect to ξ, for a given $(u^s, v, \psi, \Lambda, \tau)$ the Newton corrections $\delta u^s, \delta v, \delta\psi, \delta\Lambda, \delta\tau$ satisfy the linear $(n_0 + 2n + 2)$ system:

$$a_s\delta u^s + b_s\delta\Lambda = r_s, \tag{23}$$

$$B_s\delta u^s + A\delta v + c\delta\Lambda + d\delta\tau = r_1, \tag{24}$$

$$C_s\delta u^s + D\delta v + A\delta\psi + e\delta\Lambda + f\delta\tau = r_2, \tag{25}$$

$$(u_{1c}, \delta v) = -(u_{1c}, v), \tag{26}$$

$$(u_{1c}, \delta\psi) = -(u_{1c}, \psi), \tag{27}$$

where

$$V_u^s = V_u\big(I_0^{-1}u^s, \lambda_c + \xi\Lambda, 0\big),$$

$$V_u^t = V_u\Big(I_0^{-1}u^s + \xi u_{1c} + \xi^2 v, \lambda_c + \xi\Lambda, \xi^2\tau\bar{u}\Big), \text{ etc.,}$$

$$a_s = I_0 V_{uu}^s I_0^{-1}, \quad b_s = I_0 V_{u\lambda}^s \xi, \quad r_s = -I_0 V_u^s,$$

$$B_s = \frac{V_{uu}^t I_0^{-1}}{\xi^2}, \quad A = V_{uu}^t, \quad c = \frac{V_{u\lambda}^t}{\xi}, \quad d = V_{uw}^t \bar{u}, \quad r_1 = -\frac{V_u^t}{\xi^2},$$

$$C_s = \frac{V^t_{uuu}(u_{1c} + \xi\psi)I_0^{-1}}{\xi}, \quad D = \xi V^t_{uuu}(u_{1c} + \xi\psi), \quad e = V^t_{uu\lambda}(u_{1c} + \xi\psi),$$

$$f = \xi V^t_{uuw}(u_{1c} + \xi\psi)\bar{u}, \quad r_2 = -\frac{V^t_{uu}(u_{1c} + \xi\psi)}{\xi}. \tag{28}$$

In the practical realization one can take advantage of a partitioning technique (see, e.g., Werner and Spence, 1984; Riks et al., 1990).

Compute z_i, c_i, $r_{2+}i$, v_i, τ_i, $i = 1, 2$, h, r_5, $\delta\psi$, $\delta\Lambda$, δv, $\delta\tau$ and δu^s in

$$a_s z_1 = -b_s, \quad a_s z_2 = r_s, \tag{29}$$

$$c_1 = c + B_s z_1, \quad r_3 = r_1 - B_s z_2, \quad c_2 = e + C_s z_1, \quad r_4 = r_2 - C_s z_2, \tag{30}$$

$$\begin{pmatrix} A & d \\ u_{1c}^{\mathrm{T}} & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ \tau_1 \end{pmatrix} = -\begin{pmatrix} c_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} A & d \\ u_{1c}^{\mathrm{T}} & 0 \end{pmatrix}\begin{pmatrix} v_2 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} r_3 \\ -(u_{1c}, v) \end{pmatrix}, \tag{31}$$

$$h = c_2 + Dv_1 + \tau_1 f, \quad r_5 = r_4 - Dv_2 - \tau_2 f, \tag{32}$$

$$\begin{pmatrix} A & h \\ u_{1c}^{\mathrm{T}} & 0 \end{pmatrix}\begin{pmatrix} \delta\psi \\ \delta\Lambda \end{pmatrix} = \begin{pmatrix} r_5 \\ -(u_{1c}, \psi) \end{pmatrix}, \tag{33}$$

$$\delta v = v_1\delta\Lambda + v_2, \quad \delta\tau = \tau_1\delta\Lambda + \tau_2, \quad \delta u^s = z_1\delta\Lambda + z_2. \tag{34}$$

Direct solution of systems (31) and (33) with Gaussian elimination might require full pivoting strategy in order to avoid severe accumulation of roundoff errors. However, full pivoting destroys the bordered structure of the coefficient matrix. Let j be the largest component in u_{1c}. Both of the two systems can be solved by decomposing the same substiffness matrix $\bar{A}$ of order $n-1$ which is obtained by deleting the jth row and the jth column from A. We refer readers to Riks et al. (1990). Thus each Newton step requires the solutions of systems of equations with only the two coefficient matrices a_s and $\bar{A}$.

In solving system (22), we never use $\xi = 0$ so that even when applying Newton's method, the solution $(I_0 u^c, v_0, \psi_0, \Lambda_0, \tau_0)$ to system $H(u^s, v, \psi, \Lambda, \tau, 0) = 0$ (especially item $V^c_{uuu}u^2_{1c}$) needs not be computed. In the first continuation step starting from $\xi = 0$, we can choose $(u^s, v, \psi, \Lambda, \tau) = (I_0 u^c, 0, 0, 0, 0)$ as the initial value. In addition, the directional derivatives of the tangent stiffness matrix in the computation can be approximated by difference quotient (Wriggers and Simo, 1990).

4. Numerical examples

In this section, we present three examples to demonstrate the efficiency and applicability of the proposed method.

Example 4.1. Ziegler's two-degree-of-freedom cantilevered model.

The model shown in Fig. 1 consists of two rigid weightless links of equal length l, interconnected with each other and being supported by frictionless hinges and corresponding nonlinlearly elastic rotational

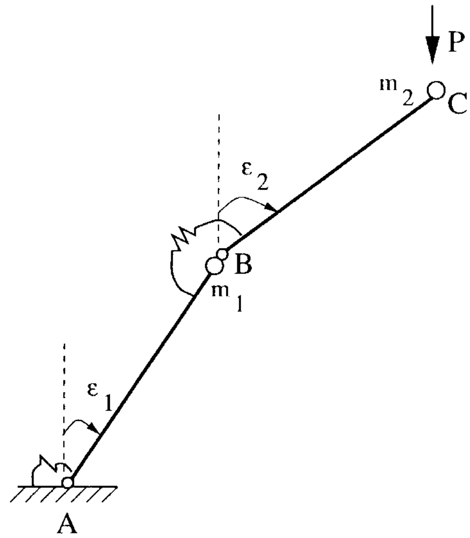

Fig. 1. Ziegler's cantilevered model subject to vertical load.

springs of quadratic type. The unstressed configuration is specified by the initial geometric imperfections ε_1 and ε_2. The model at its top end is subjected to a vertical load P.

The nondimensional total potential energy of the system is given by

$$V(\theta_1, \theta_2, \lambda, \varepsilon_1, \varepsilon_2) = \frac{1}{2}\theta_1^2 + \frac{1}{3}\delta_1\theta_1^3 + \frac{1}{2}(\theta_2 - \theta_1)^2 + \frac{1}{3}\delta_2(\theta_2 - \theta_1)^3 \\ - \lambda\left[\cos\varepsilon_1 - \cos(\theta_1 + \varepsilon_1) + \cos\varepsilon_2 - \cos(\theta_2 + \varepsilon_2)\right] \tag{35}$$

where θ_1 and θ_2 are the incremental angles of deformation of the system ($u{:=}(\theta_1, \theta_2)^t$), k is the linear spring component common for both springs and $\delta_i (i = 1, 2)$ are the nonlinear components of the corresponding quadratic springs, $\delta_i > 0(< 0)$ express the corresponding spring is of hard (soft) type, $\lambda{:=}Pl/k$. Let the imperfection vector

$$w = (\varepsilon_1, \varepsilon_2)^t = \varepsilon\bar{u}, \quad \bar{u}{:=}(a, b)^t, \quad a^2 + b^2 = 1. \tag{36}$$

Eqns (1) and (2) for the model become

$$\theta_1 + \delta_1\theta_1^2 - \theta_2 + \theta_1 - \delta_2(\theta_2 - \theta_1)^2 - \lambda\sin(\theta_1 + \varepsilon_1) = 0, \theta_2 - \theta_1 + \delta_2(\theta_2 - \theta_1)^2 - \lambda\sin(\theta_2 + \varepsilon_2) = 0 \tag{37}$$

and

$$\left[2 + 2\delta_1\theta_1 - 2\delta_2(\theta_1 - \theta_2) - \lambda\cos(\theta_1 + \varepsilon_1)\right]\phi_1 - \left[1 + 2\delta_2(\theta_2 - \theta_1)\right]\phi_2 = 0,$$

$$-\left[1 + 2\delta_2(\theta_2 - \theta_1)\right]\phi_1 + \left[1 + 2\delta_2(\theta_2 - \theta_1) - \lambda\cos(\theta_2 + \varepsilon_2)\right]\phi_2 = 0 \tag{38}$$

where $\phi = (\phi_1, \phi_2)^t$.

In the following numerical computations, we take $\delta_1 = -2.5$, $\delta_2 = -0.75$. At the first bifurcation point of the perfect model, the buckling load $\lambda_c = 0.38197$ and corresponding buckling mode $u_{1c} = (\theta_{1c}, \theta_{2c})^t$

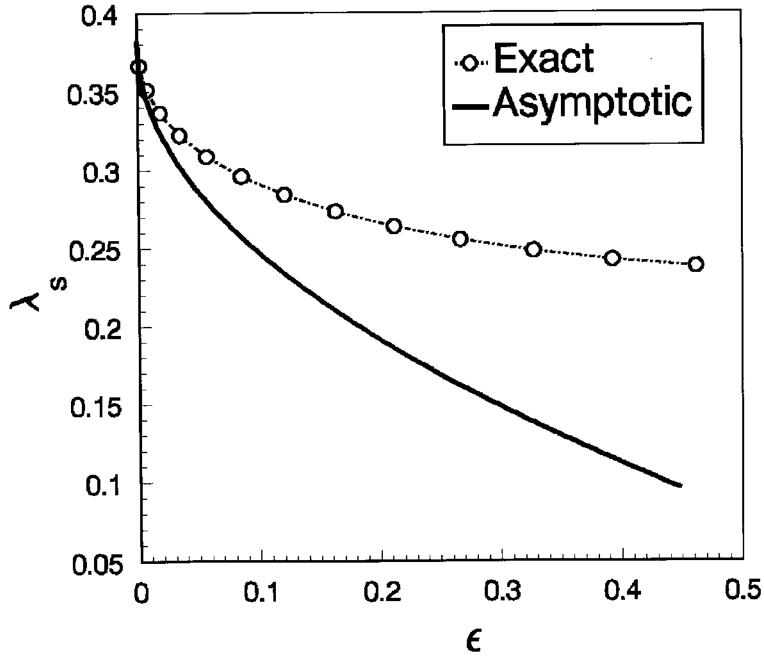

Fig. 2. Dependence of the buckling load on the imperfection amplitude for the imperfection mode (−0.707, 0.707).

$= (0.52573, 0.85065)^t$. This is an asymmetric bifurcation point. Letting $(\theta_1, \theta_2)^t = \xi(\theta_{1c}, \theta_{2c})^t + \xi^2(v_1, v_2)^t$, $\lambda = \lambda_c + \xi\Lambda$, $(\varepsilon_1, \varepsilon_2)^t = \xi^2\tau(a, b)^t$, $\phi = (\theta_{1c}, \theta_{2c})^t + \xi(\psi_1, \psi_2)^t$ and defining the inner product $(u, v) = u_1v_1 + u_2v_2$, we can obtain extended system (7) for the present problem. By using the continuation method described in Section 3, we can determine $v(\xi)$, $\psi(\xi)$, $\Lambda(\xi)$ and $\tau(\xi)$ from the extended system. The numerical results (exact) for $(a, b) = (-0.707, 0.707)$ is plotted in Fig. 2. For comparison, the asymptotic solution [up to $O(\varepsilon)$] based on Wu and Wang (1997) is also shown.

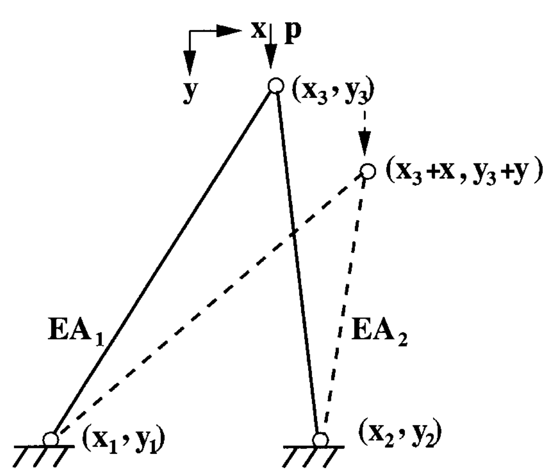

Fig. 3. Two-bar non-shallow arch under a vertical load.

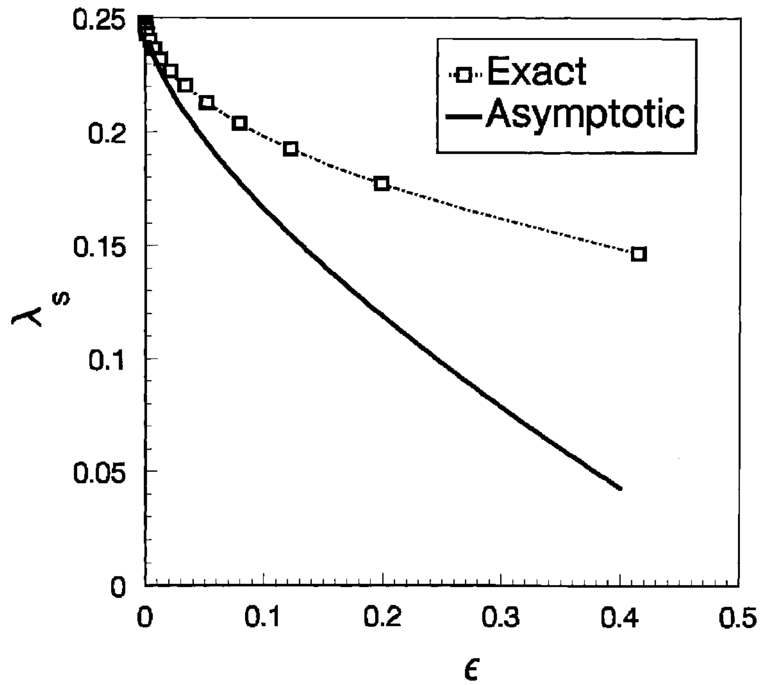

Fig. 4. Dependence of the buckling load on the imperfection amplitude for the critical imperfection mode.

Example 4.2. A two-bar non-shallow arch subjected to vertical load at top node.

This model (see Fig. 3) is investigated by Ikeda and Murota (1990) for determining the critical imperfection of the arch. The nondimensional potential energy of the arch is given by

$$V(u, \lambda, \bar{u}) = \frac{1}{2}a_1\left(\frac{\hat{l}_1 - l_1}{l_1}\right)^2 + \frac{1}{2}a_2\left(\frac{\hat{l}_2 - l_2}{l_2}\right)^2 - \lambda y \tag{39a}$$

where

$$u = (x, y)^t, \quad \lambda = \frac{P}{EA},$$

$$w = (\Delta x_1, \Delta y_1, \Delta x_2, \Delta y_2, \Delta x_3, \Delta y_3, \Delta a_1, \Delta a_2)^t,$$

$$(x_1, y_1, x_2, y_2, x_3, y_3, a_1, a_2)^t = (-1, 3, 1, 3, 0, 0, 1, 1)^t + w,$$

$$l_i = \left\{(x_3 - x_i)^2 + (y_3 - y_i)^2\right\}^{1/2},$$

$$\hat{l}_i = \left\{(x + x_3 - x_i)^2 + (y + y_3 - y_i)^2\right\}^{1/2}, \quad i = 1, 2. \tag{39b}$$

The imperfection vector w is written as

$$w = \varepsilon d, \quad \sum_{i=1}^{8} d_i^2 = 1. \tag{40}$$

It is easy to check that, for the perfect structure, there exists invariant subspace $U_0 := \{(0, y)^t, y \in R\}$ of R^2; in addition, it has no trivial symmetric fundamental path. Using the method described in Section 3.2, we get the buckling load $\lambda_c = 0.24776$ and corresponding buckling mode $u_{1c} = (x_{1c}, y_{1c})^t = (0, 1)^t$ of the perfect model. This is a symmetric bifurcation point. Using the method described in Sections 3.2 and 3.3, we obtain the load drop (exact) vs imperfection amplitude curve (Fig. 4) for the critical imperfection mode (Ikeda and Murota, 1990)

$$d = (-0.28404, -0.26061, -0.28404, 0.26061, 0.56812, 0, 0.43592, -0.43592)^t. \tag{41}$$

The asymptotic solution [up to $O(\varepsilon)$] derived from Wu and Wang (1997) is also shown for comparison.

Example 4.3. A simply supported beam on a nonlinear softening elastic foundation subjected to axial load (Fig. 5).

The nondimensional potential energy of the beam is given by (Potier-Ferry, 1987):

$$\Phi(u, \lambda, w) = \int_0^l \left(\frac{1}{2} u''^2 - \frac{1}{2} \lambda u'^2 + \frac{1}{2} u^2 - \frac{1}{4} u^4 - \lambda w' u' \right) \mathrm{d}x \tag{42}$$

where u, w, l, and λ are, respectively, nondimensional incremental lateral displacement, initial geometric imperfection and length of the beam, and axial load. For illustration, we take $l = \pi$ in the following discussions. The beam is assumed to be simply supported.

We choose a uniform mesh size, $h = \pi/(N+1)$, N: integer, and nodes $x_i = ih$, $i = 1, 2, \ldots, N$. Starting from (42) and using difference-variation (evaluate the integral with trapezoid formula and use central difference quotient approximations to u'' and u') discretization to it leads to the corresponding finite-dimensional potential energy function V:

$$\begin{aligned} V(u, \lambda, w) = {} & \frac{1}{2} h \sum_1^N \left(\frac{u_{i-1} - 2u_i + u_{i-1}}{h^2} \right)^2 - \frac{1}{2} h\lambda \left[\frac{1}{2} \frac{u_1^2}{h^2} + \sum_1^N \left(\frac{u_{i+1} - u_{i-1}}{2h} \right)^2 + \frac{1}{2} \frac{u_N^2}{h^2} \right] + \frac{h}{2} \left(\sum_1^N u_i^2 \right) \\ & - \frac{h}{4} \left(\sum_1^N u_i^4 \right) - \lambda h \left[\frac{1}{2} \frac{w_1}{h} \frac{u_1}{h} + \sum_1^N \left(\frac{w_{i+1} - w_{i-1}}{2h} \right) \left(\frac{u_{i+1} - u_{i-1}}{2h} \right) + \frac{1}{2} \frac{w_N}{h} \frac{u_N}{h} \right] \end{aligned} \tag{43}$$

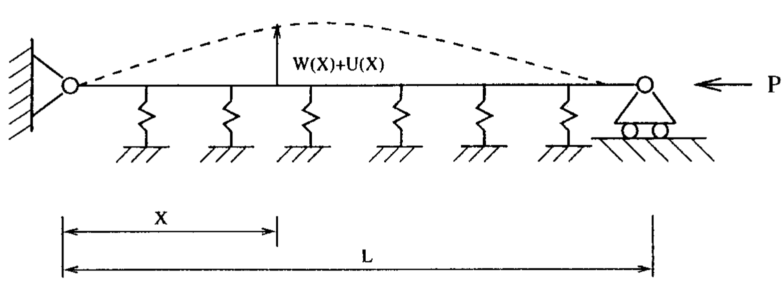

Fig. 5. A beam on a softening elastic foundation under axial compression load.

where $u_1 = u_{N+1} = w_1 = w_{N+1} = 0$, $u = (u_1, u_2, \dots, u_N)^{\mathrm{T}}$ and $w = (w_1, w_2, \dots, w_N)^{\mathrm{T}}$ are, respectively, the column vectors of the nodes of the incremental lateral displacement and initial geometric imperfection. In the above finite-dimensional system, we define the inner product of u and v as follows:

$$(u, v) = h\left(\sum_{1}^{N} u_i v_i\right). \tag{44}$$

Let $N = 40$. At the first bifurcation point of the corresponding perfect beam (with the fundamental path $u_0(\lambda) \equiv 0$), $\lambda_c = 2.002939$, and the buckling mode u_{1c} is normalized according to the norm $\|u_{1c}\| = 1$ and its expression is omitted. This is also a symmetric bifurcation point. In the computation of the buckling strength, we consider the initial geometric imperfection which has the shape of classical buckling mode u_{1c}, and let the imperfection

$$w = \varepsilon u_{1c} \tag{45}$$

Based on the method of Section 3, the dependence of the buckling load λ_s (exact) on the maximum amplitude $u_0^{\max}(:=\varepsilon u_{1c}^{\max}$ where $u_{1c}^{\max}$ is the largest component in the normalized buckling mode u_{1c}) of initial geometric imperfection is shown in Fig. 6. For comparision, the asymptotic solution [up to $O(\varepsilon)$] based on Wu and Wang (1997) is also shown.

From Figs. 2, 4 and 6, one finds that, based on the proposed direct method, the exact buckling load for sufficiently large imperfection amplitude can be achieved. Furthermore, it is evident that the asymptotic analysis is a reasonable approximation only for small range of the imperfection amplitude.

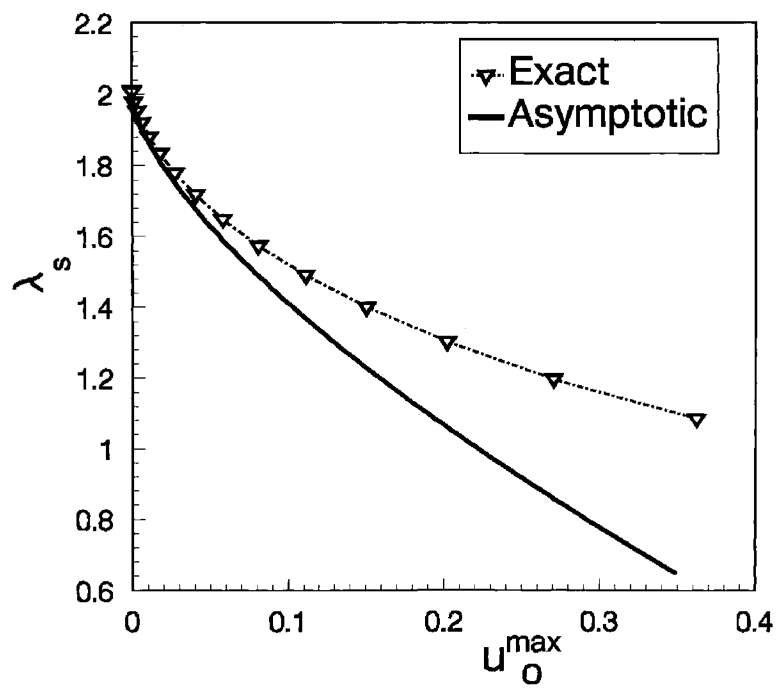

Fig. 6. Dependence of the buckling load on the maximum initial geometric imperfection amplitude.

5. Conclusions

An extended system method for determining the buckling strength of imperfect structures has been introduced, under the assumptions of a symmetric or a linear fundamental path and existence of a simple symmetry-breaking or a simple bifurcation point. The solution to the extended system can be regularized by ξ, the amplitude of projection of incremental displacement u from the fundamental path on the normalized buckling mode u_{1c} of the perfect structure. Thus, standard methods can be used to directly compute the buckling load of imperfect structures by continuing ξ with the extended system. The implementation of Newton's method solving the extended system—a partitioning procedure has also been given.

This method can allow simultaneous treatment of geometric and material imperfections, determine the exact limit point load and corresponding displacement without tracking the equilibrium paths of the imperfect structures, permit results to be implemented in a FEM program. The three numerical examples have shown that one can get the exact buckling load for sufficiently large imperfection amplitude and the algorithm has good convergence.

Acknowledgements

The author would like to thank Prof. W. B. Krätzig for his advice. This work is supported by the Alexander von Humboldt Foundation of Germany and the State Education Commission of P.R. China.

References

Budiansky, B., 1974. Theory of buckling and post-buckling behavior of elastic structure. In: Yih, C.-S. (Ed.), Advances in Applied Mechanics, vol. 14. Academic Press, New York, pp. 1–65.

Casciaro, R., Salerno, G., Lanzo, A.D., 1992. Finite element asymptotic analysis of slender elastic structure: a simple approach. Int. J. Numer. Methods Engrg 35, 1397–1426.

El Naschie, M.S., 1990. Stress, Stability and Chaos in Structural Engineering: An Energy Approach. McGraw-Hill, London.

Eriksson, A., 1994. Fold lines for sensitivity analyses in structural instability. Comput. Methods Appl. Mech. Engrg 114, 77–101.

Godoy, L.A., Barbero, E.J., Raftoyiannis, I.G., 1995. Finite elements for post-buckling analysis. I. The W-formulation. Comput. Struct. 56 (6), 1009–1017.

Ikeda, K., Murota, K., 1990. Critical initial imperfection of structures. Int. J. Solids Structures 26 (8), 865–886.

Koiter, W.T., 1945. On the Stability of Elastic Equilibrium. Dissertation, Delft University of Technology, The Netherlands. English translation: NASA Rep. TTF-10: 833 (1967).

Peek, R., Kheyrkhahan, M., 1993. Postbuckling behavior and imperfection sensitivity of elastic structures by the Lyapunov–Schmidt–Koiter approach. Comput. Methods Appl. Mech. Engrg 108, 261–279.

Potier-Ferry, M., 1987. Foundations of elastic post-buckling theory. In: Arbocz, J., Potier-Ferry, M., Singer, J., Tvergaard, V. (Eds.), Buckling and Post-buckling, Lectures Notes in Physics, vol. 288. Springer-Verlag, Berlin, pp. 83–142.

Riks, E., 1979. An incremental approach to the solution of snapping and buckling problems. Int. J. Solids Structures 15 (7), 529–551.

Riks, E., Borgan, F.A., Rankin, C.C., 1990. Numerical aspects of shell stability analysis. In: Krätzig, W.B., Oñate, E. (Eds.), Computational Mechanics of Nonlinear Response of Shell. Springer, Berlin, pp. 125–151.

Seydel, R., 1979. Numerical computation of branch points in nonlinear equations. Numer. Math. 33, 339–352.

Thompson, J.M.T., Hunt, G.W., 1973. A General Theory of Elastic Stability. Wiley, London.

Triantafyllidis, N., Peek, R., 1992. On stability and the worst imperfection shape in solids with nearly simultaneous eigenmodes. Int. J. Solids Structures 29 (18), 2281–2299.

Troger, H., Steindl, A., 1991. Nonlinear Stability and Bifurcation Theory. Springer, New York.

Werner, B., 1984. Regular systems for bifurcation problem with underlying symmetries. In: Küpper, T., Mittelmann, H.D., Weber, H. (Eds.), Numerical Methods for Bifurcation Problems, ISNM, 70. Birkhäuser, Basel, pp. 562–574.

Werner, B., Spence, A., 1984. The computation of symmetry-breaking bifurcation points. SIAM J. Numer. Anal. 21 (2), 388–399.

Wriggers, P., Simo, J.C., 1990. A general procedure for the direct computation of turning and bifurcation points. Int. J. Numer. Methods Engrg 30, 155–176.

Wu, B., 1995. Numerical nonlinear analysis of secondary buckling in stability problems, Comput. Methods Appl. Mech. Engrg 120, 183–193.

Wu, B., Wang, Z., 1997. A perturbation method for the determination of the buckling strength of imperfection-sensitive structures. Comput. Methods Appl. Mech. Engrg 145, 203–215.